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On Smoothness Characterized by Bernstein Type Operators*

DING-XUAN ZHOU

Institute of Mathematics, Academia Sinica, Beijing, 100080, People's Republic of China; and Department of Mathematics, Zhejiang University, Advanced Institute of Mathematics, Hangzhou, Zhejiang, 310027, People's Republic of China

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We characterize the higher orders of smoothness of functions in $C[0,1]$ by Bernstein polynomials and Kantorovich operators. This task is carried out by means of the rate of convergence for combinations of these operators and the behavior of their derivatives.

1. INTRODUCTION

The Bernstein polynomials on $C[0,1]$ are given by

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \equiv \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x). \tag{1.1}$$

It was shown by H. Berens and G. G. Lorentz [3] in 1972 that if $0<\alpha<2$ then

$$|B_n(f,x)-f(x)| \leqslant M(x(1-x)/n)^{\alpha/2}$$

if and only if

$$\|\Delta_h^2 f\|_{C[h,1-h]} \equiv \|f(x+h)-2f(x)+f(x-h)\|_{C[h,1-h]} = O(h^\alpha).$$

Many expressions concerning the connection between the rate of convergence for Bernstein polynomials and the smoothness of functions were

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explored since then. In fact, for $0 \leqslant \eta < 2$, $0 < \alpha < 2$, $\varphi(x) = x(1-x)$, we have

$$\|(\varphi(x))^{-\eta/2}(B_n(f, x) - f(x))\|_{C[0,1]} = O(n^{-\alpha/2})$$
$$\Leftrightarrow \|(\varphi(x))^{(\alpha-\eta)/2} \Delta_h^2 f(x)\|_{C[h, 1-h]} = O(h^\alpha). \tag{1.2}$$

As mentioned above, the case $0 < \alpha = \eta < 2$ was proved by H. Berens and G. G. Lorentz and reproved in 1978 by M. Becker [1]. Most parts of the proof of (1.2) are due to Z. Ditzian. In 1979, he proved the case $\eta = 0$ in [5] which was reproved in 1984 by V. Totik in [16]. Ditzian also showed the case $0 < \eta \leqslant \alpha < 2$ in 1980 in [6]. Then, in 1987 he proved the case $0 < \alpha < \eta < 2$, $\alpha + \eta \leqslant 2$ in [8] and meanwhile posed a conjecture for the final case $0 < \alpha < \eta < 2$, $\alpha + \eta > 2$. Finally, in 1992 the author proved this final case of (1.2) in [18] by confirming the conjecture of Z. Ditzian.

In all the statements of (1.2), two cases are of most importance. The first one is $\eta = 0$. The task of such a result is to characterize the classes of functions which have some given orders of approximation for a sequence of approximating operators. For the Bernstein type operators, this task is often implemented by means of the so-called Ditzian-Totik moduli of smoothness (see [5, 9, 10, 16, 17]). These new moduli of smoothness can also be used for the characterization of orders of approximation by algebraic polynomials, see [10]. For a recent contribution along this line, see the monograph of Z. Ditzian and V. Totik [10].

The second important case of (1.2) is $\eta = \alpha$. The aim of such a result is to characterize the smoothness of functions such as Lipschitz smoothness by means of the rate of convergence for some classes of approximating operators. In [1, 2, 3] it was shown that the Lipschitz functions of one and second orders can be characterized by means of Bernstein type operators which reproduce linear functions. Then, it is natural to investigate whether such a characterization can be extended to higher orders of smoothness and also to the Bernstein type operators which do not reproduce linear functions such as Kantorovich operators. The first purpose of this paper is to give such a characterization.

Since the Bernstein polynomials cannot be used for the investigation of higher orders of smoothness, P. L. Butzer [4] introduced the combinations of Bernstein polynomials which have higher orders of approximation. Z. Ditzian and V. Totik [10, p. 116] (see also [5, p. 278]) extended this method of combinations and defined for the operators $\{L_n(f, x)\}_{n \in N}$ the combination $L_n(f, r, x)$ as

$$L_n(f, r, x) = \sum_{i=0}^{r-1} c_i(n) L_{n_i}(f, x), \tag{1.3}$$

where with an absolute constant $C \in N$, n_i and $c_i(n)$ satisfy

$$\begin{aligned}
&(a)\quad n = n_0 < \cdots < n_{r-1} \leqslant Cn;\\
&(b)\quad \sum_{i=0}^{r-1} |c_i(n)| \leqslant C;\\
&(c)\quad \sum_{i=0}^{r-1} c_i(n) = 1;\\
&(d)\quad \sum_{i=0}^{r-1} c_i(n)\, n_i^{-k} = 0, \qquad \text{for} \quad k = 1, ..., r-1.
\end{aligned} \tag{1.4}$$

By means of these combinations we can extend the Berens-Lorentz Theorem to higher orders of smoothness. In fact, we show in this paper that for $0 < \alpha < r$, $f \in C[0, 1]$

$$\begin{aligned}
&\omega_r(f, t) = O(t^\alpha)\\
&\quad \Leftrightarrow |B_n(f, r-1, x) - f(x)| \leqslant M(x(1-x)/n + n^{-2})^{\alpha/2}, \qquad r \geqslant 2 \quad (1.5)\\
&\quad \Leftrightarrow |K_n(f, r, x) - f(x)| \leqslant M'(x(1-x)/n + n^{-2})^{\alpha/2}, \qquad r \geqslant 1. \quad (1.6)
\end{aligned}$$

Here $K_n(f, x)$ is the Kantorovich operator given by

$$K_n(f, x) = \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt P_{n,k}(x) \tag{1.7}$$

and $\omega_r(f, t)$ is the rth classical modulus of smoothness defined as

$$\begin{aligned}
\omega_r(f, t) &= \sup_{0 < h \leqslant t} \|\Delta_h^r f(x)\|_{C[rh/2,\, 1-rh/2]},\\
\Delta_h^r f(x) &= \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + (k - r/2)\, h).
\end{aligned} \tag{1.8}$$

We note that the case $r = 2$ in (1.5) is just the Berens-Lorentz Theorem in [3].

The second purpose of this paper is to consider the close connection between the derivatives of the Bernstein type operators and the smoothness of functions which has been investigated by Z. Ditzian, V. Totik, K. G. Ivanov and some other mathematicians (see [7, p. 25], [9, p. 87] and [10, Chapter 7]). We extend a result of Ditzian [7] to higher orders of smoothness and will show for $0 < \alpha < r$ that

$$\omega_r(f, t) = O(t^\alpha) \Leftrightarrow |B_n^{(r)}(f, x)| \leqslant M\left(\min\left\{n^2, \frac{n}{x(1-x)}\right\}\right)^{(r-\alpha)/2} \tag{1.9}$$

under the assumption that $\omega_r(f, t) = O(t^\beta)$ with certain $\beta > 0$ and

$$\omega_r(f, t) = O(t^\alpha) \Leftrightarrow |K_n^{(r)}(f, x)| \leqslant M' \left(\min\left\{n^2, \frac{n}{x(1-x)}\right\}\right)^{(r-\alpha)/2}. \quad (1.10)$$

Let us mention that all the results here are also valid for the Bernstein-Durrmeyer operators given in [9, 11].

2. Direct Theorems

The direct part of (1.5) and (1.6) is the following

THEOREM 2.1. *Let $f \in C[0, 1]$. Then we have for $r \geqslant 2$*

$$|B_n(f, r-1, x) - f(x)| \leqslant M\omega_r(f, \sqrt{x(1-x)/n + n^{-2}}) \quad (2.1)$$

and for $r \geqslant 1$

$$|K_n(f, r, x) - f(x)| \leqslant M'\omega_r(f, \sqrt{x(1-x)/n + n^{-2}}), \quad (2.2)$$

where M and M' are constants independent of n and x.

We recall that $K_r(f, t^r)$, the Peetre K-functional given by

$$K_r(f, t^r) = \inf_{g^{(r-1)} \in A.C._{loc}} \{\|f - g\|_\infty + t^r \|g^{(r)}\|_\infty\} \quad (2.3)$$

is equivalent to $\omega_r(f, t)$, see [10, p. 11]. That is,

$$M_0^{-1}\omega_r(f, t) \leqslant K_r(f, t^r) \leqslant M_0\omega_r(f, t) \quad (2.4)$$

with a constant M_0 independent of $1 \geqslant t > 0$ and $f \in C[0, 1]$.

Using the expressions of the moments of the Bernstein polynomials and Kantorovich operators given in [10, p. 134] we have for $k = 1, ..., r-1$

$$B_n((\cdot - x)^k, r-1, x) = 0,$$
$$K_n((\cdot - x)^k, r, x) = 0$$

and for $r \in N$

$$\begin{aligned} B_n((\cdot - x)^{2r}, x) &\leqslant M_1(x(1-x)/n + n^{-2})^r, \\ K_n((\cdot - x)^{2r}, x) &\leqslant M_1(x(1-x)/n + n^{-2})^r. \end{aligned} \quad (2.5)$$

Then we can prove Theorem 2.1 by the standard method for direct theorems (see [3, p. 699], [5, p. 284] and [10, Theorem 9.3.2]). We omit the detailed proof here.

The direct part of (1.9) and (1.10) is the following

THEOREM 2.2. *Let $f \in C[0,1]$, $r \in N$, $L_n(f,x)$ given by (1.1) or (1.7). Then we have*

$$|L_n^{(r)}(f,x)| \leq M\left(\min\left\{\sqrt{\frac{n}{x(1-x)}}, n\right\}\right)^r \omega_r\left(f, \left(\min\left\{\sqrt{\frac{n}{x(1-x)}}, n\right\}\right)^{-1}\right), \tag{2.6}$$

where M is a constant independent of n and x.

Proof of Theorem 2.2. From the expressions of the derivatives of $L_n(f,x)$ given by [10, (9.4.3) and (9.4.8)] we know that

$$\|\varphi^{r/2} L_n^{(r)}(f)\|_\infty \leq M_2 n^{r/2} \|f\|_\infty, \quad \text{if } f \in C[0,1]; \tag{2.7}$$

$$\|L_n^{(r)}(f)\|_\infty \leq M_2 n^r \|f\|_\infty, \quad \text{if } f \in C[0,1]; \tag{2.8}$$

$$\|L_n^{(r)}(f)\|_\infty \leq M_2 \|f^{(r)}\|_\infty, \quad \text{if } f^{(r-1)} \in A.C._{loc}, \tag{2.9}$$

where M_2 is a constant independent of n and f. Thus, by taking infimum over g we have

$$\begin{aligned}
|L_n^{(r)}(f,x)| &\leq \inf\{|L_n^{(r)}(f-g,x)| + |L_n^{(r)}(g,x)| : g^{(r-1)} \in A.C._{loc}\} \\
&\leq \inf\left\{\min\left\{M_2\left(\frac{n}{x(1-x)}\right)^{r/2} \|f-g\|_\infty, M_2 n^r \|f-g\|_\infty\right\}\right. \\
&\quad \left. + M_2 \|g^{(r)}\|_\infty : g^{(r-1)} \in A.C._{loc}\right\} \\
&\leq M_2\left(\min\left\{\sqrt{\frac{n}{x(1-x)}}, n\right\}\right)^r K_r\left(f, \left(\min\left\{\sqrt{\frac{n}{x(1-x)}}, n\right\}\right)^{-r}\right).
\end{aligned}$$

By (2.4), we know that (2.6) holds with a constant M independent of n and x. The proof of Theorem 2.2 is complete.

Thus, we have proved the direct estimates stated in section 1. In the following two sections we shall show the corresponding inverse results respectively.

3. INVERSE RESULTS FOR RATE OF CONVERGENCE

In this section we give the inverse part of (1.5) and (1.6). We use the elementary method for inverse problems developed by M. Becker [1], H. Berens and G. G. Lorentz [3], Z. Ditzian and V. Totik [10, Chapter 9].

THEOREM 3.1. *Let $f \in C[0, 1]$, $r \in N$, $0<\alpha<r$. Then we have*

$$\begin{aligned}
&\omega_r(f, h) = O(h^\alpha) \\
&\Leftrightarrow |B_n(f, r-1, x) - f(x)| \leqslant M\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2}, \quad \text{if } r \geqslant 2 \qquad (3.1) \\
&\Leftrightarrow |K_n(f, r, x) - f(x)| \leqslant M'\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\alpha/2}, \quad \text{if } r \geqslant 1. \qquad (3.2)
\end{aligned}$$

Remark. The term $x(1-x)/n + n^{-2}$ can not be replaced by $x(1-x)/n$ for $r \geqslant 3$ in (3.1) and for $r \geqslant 1$ in (3.2). Also, for $r > 2$, r can not be replaced by $2r$ in (3.1). The exact calculations in [5, p. 279] and [10, p. 2] show that this can not be improved much. Combining with the Berens-Lorentz Theorem in [3] we can see the similarity between the Bernstein polynomials and the best polynomials of approximation. In fact, we know from [10, 14, 15] that for $f \in C[-1, 1]$, $r \in N$, $0 < \alpha < r$, $\omega_r(f, h) = O(h^\alpha)$ is equivalent to the existence of a sequence of polynomials $\{P_n\}$ of degree n that satisfies

$$|f(x) - P_n(x)| \leqslant M(\sqrt{1-x^2}/n + n^{-2})^\alpha.$$

Moreover, the term $\sqrt{1-x^2}/n + n^{-2}$ can be replaced by $\sqrt{1-x^2}/n$ if and only if $r = 1, 2$.

Proof of Theorem 3.1. We only prove the inverse part of (3.1) here since the proof of (3.2) is the same.

We denote

$$d(n, x, t) = \max\left\{\frac{1}{n}, \max_{0 \leqslant k \leqslant r}\left\{\sqrt{\frac{\varphi(x + (r/2 - k)\, t)}{n}}\right\}\right\}.$$

Let $0 < t \leqslant h < 1/(8r)$, $x \pm rt/2 \in (0, 1)$, $n \in N$. We have

$$\begin{aligned}
|\Delta_t^r f(x)| &\leqslant |\Delta_t^r(f - B_n(f, r-1, \cdot))(x)| \\
&\quad + \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left|B_n^{(r)}\left(f, r-1, x + \sum_{j=1}^r y_j\right)\right| dy_1 \cdots dy_r \\
&\leqslant 4^r M(d(n, x, t))^\alpha \\
&\quad + \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left|B_n^{(r)}\left(f_d, r-1, x + \sum_{j=1}^r y_j\right)\right| dy_1 \cdots dy_r \\
&\quad + \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left|B_n^{(r)}\left(f - f_d, r-1, x + \sum_{j=1}^r y_j\right)\right| dy_1 \cdots dy_r \\
&:= I + J_1 + J_2.
\end{aligned}$$

Here, by (2.4), f_d is chosen over $d>0$ such that

$$\|f_d^{(r)}\|_\infty \leqslant 2d^{-r}K_r(f, d^r) \leqslant 2M_0\, d^{-r}\omega_r(f, d), \tag{3.3}$$

$$\|f-f_d\|_\infty \leqslant 2K_r(f, d^r) \leqslant 2M_0\,\omega_r(f, d). \tag{3.4}$$

We need to estimate J_1 and J_2.

By (2.9) and (3.3) we have

$$\begin{aligned} J_1 &\leqslant \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \sum_{i=0}^{r-2} |c_i(n)|\ \|B_{n_i}^{(r)}(f_d)\|_\infty\, dy_1 \cdots dy_r \\ &\leqslant CM_2\, \|f_d^{(r)}\|_\infty\, t^r \leqslant 2CM_2 M_0\, d^{-r}t^r\omega_r(f, d). \end{aligned} \tag{3.5}$$

By (2.8) and (3.4) we have

$$\begin{aligned} J_2 &\leqslant \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \sum_{i=0}^{r-2} |c_i(n)|\ M_2 n_i^r\, \|f-f_d\|_\infty\, dy_1 \cdots dy_r \\ &\leqslant C^{r+1}M_2 n^r t^r\, \|f-f_d\|_\infty \leqslant 2C^{r+1}M_2 M_0 n^r t^r \omega_r(f, d). \end{aligned} \tag{3.6}$$

By (2.7) and (3.4) we also have

$$\begin{aligned} J_2 \leqslant{}& \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \sum_{i=0}^{r-2} |c_i(n)|\ M_2 n_i^{r/2}\, \|f-f_d\|_\infty \\ &\times \left(\varphi\left(x+\sum_{j=1}^{r} y_j\right)\right)^{-r/2} dy_1 \cdots dy_r \\ \leqslant{}& 2M_0 C^{r/2+1} M_2 C_r t^r \omega_r(f, d) \left(\max_{0 \leqslant k \leqslant r} \{\sqrt{\varphi(x+(r/2-k)\,t)/n}\}\right)^{-r}. \end{aligned} \tag{3.7}$$

Here we have used inequality (3.8) which we shall give and prove in the following Lemma 3.2.

Thus, by taking infimum over (3.6) and (3.7), we have

$$\begin{aligned} |\Delta_t^r f(x)| \leqslant{}& 4^r M(d(n, x, t))^\alpha + 2CM_2 M_0\, d^{-r}t^r\omega_r(f, d) \\ &+ 2M_0 M_2 (C^{r+1} + C_r C^{r/2+1})\, t^r (d(n, x, t))^{-r}\, \omega_r(f, d). \end{aligned}$$

Let $d = d(n, x, t)$. We have

$$|\Delta_t^r f(x)| \leqslant M_3\{(d(n, x, t))^\alpha + (t/d(n, x, t))^r\, \omega_r(f, d(n, x, t))\}$$

with a constant M_3 independent of n, x, t and h.

The sequence $d(n, x, t)$ decreases to zero as n tends to infinity, also $d(n+1, x, t) \leqslant d(n, x, t) \leqslant 2d(n+1, x, t)$. Hence, for any $\delta \in (0, 1/(8r))$, there is an $n \in N$ such that $d(n, x, t) \leqslant \delta < 2d(n, x, t)$. Consequently

$$|\Delta_t^r f(x)| \leqslant 2^r M_3\{\delta^\alpha + (h/\delta)^r\, \omega_r(f, \delta)\}$$

and

$$\omega_r(f, h) \leqslant 2^r M_3\{\delta^\alpha + (h/\delta)^r \omega_r(f, \delta)\},$$

which implies $\omega_r(f, h) = O(h^\alpha)$ by [3, p. 696] and [10, Lemma 9.3.4]. Our proof of Theorem 3.1 is then complete after we prove the following Lemma 3.2.

LEMMA 3.2. *Let* $r \in N$, $0 < t < 1/8r$, $x \pm rt/2 \in (0, 1)$. Then there holds

$$\int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left(\varphi\left(x + \sum_{j=1}^{r} y_j\right)\right)^{-r/2} dy_1 \cdots dy_r \leqslant C_r \left(\max_{0 \leqslant k \leqslant r} \left\{\varphi\left(x + \left(\frac{r}{2} - k\right) t\right)\right\}\right)^{-r/2} t^r, \tag{3.8}$$

where C_r *is a constant independent of x and t.*

Proof of Lemma 3.2. The case $r = 2$ was proved in [1].
Note that $\varphi(z) = \varphi(1 - z)$ for $z \in [0, 1]$. We can assume that $x \leqslant \frac{1}{2}$.
If $\frac{1}{2} \geqslant x \geqslant ((r+1)/2)\, t$. Then for $z \in [-rt/2, rt/2]$

$$\varphi(x + z) \geqslant \frac{7}{16} \frac{x}{r+1}.$$

Hence

$$\begin{aligned}
\int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \left(\varphi\left(x + \sum_{j=1}^{r} y_j\right)\right)^{-r/2} dy_1 \cdots dy_r
&\leqslant \left(\frac{7}{16} \frac{x}{r+1}\right)^{-r/2} t^r = \left(\frac{16(r+1)}{7}\right)^{r/2} t^r x^{-r/2} \\
&\leqslant \left(\frac{16(r+1)}{7}\right)^{r/2} t^r \left(\frac{1}{2} \max_{0 \leqslant k \leqslant r} \left\{x + \left(\frac{r}{2} - k\right) t\right\}\right)^{-r/2} \\
&\leqslant (5(r+1))^{r/2} \left(\max_{0 \leqslant k \leqslant r} \left\{\varphi\left(x + \left(\frac{r}{2} - k\right) t\right)\right\}\right)^{-r/2} t^r.
\end{aligned}$$

Therefore, in the case $\frac{1}{2} \geqslant x \geqslant ((r+1)/2)\, t$, (3.8) holds if we take $C_r = (5(r+1))^{r/2}$.

The proof of the case $rt/2 < x < (r+1)\, t/2$ is somewhat different.

We first prove for $r=2m$ with $m\in N$. In this case, by (3.8) with $r=2$, we have

$$\begin{aligned}
&\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left(\varphi\left(x+\sum_{j=1}^{r}y_j\right)\right)^{-r/2}dy_1\cdots dy_r\\
&\leqslant\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left(x+\sum_{j=1}^{r}y_j\right)^{-r/2}\left(\frac{16}{7}\right)^{r/2}dy_1\cdots dy_r\\
&\leqslant\left(\frac{16}{7}\right)^{r/2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\prod_{i=1}^{m}(x+y_{2i-1}+y_{2i})^{-1}\,dy_1\cdots dy_r\\
&\leqslant\left(\frac{16}{7}\right)^{r/2}\prod_{i=1}^{m}\left\{\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}(\varphi(x+y_{2i-1}+y_{2i}))^{-1}\,dy_{2i-1}\,dy_{2i}\right\}\\
&\leqslant\left(\frac{16}{7}\right)^{r/2}\prod_{i=1}^{m}\{C_2t^2(\max_{0\leqslant j\leqslant 2}\{\varphi(x+(1-j)\,t)\})^{-1}\}\\
&\leqslant\left(\frac{16}{7}\right)^{r}C_2^m t^r(x+t)^{-m}\\
&\leqslant\left(\frac{16}{7}\right)^{r}C_2^m t^r 2^m\left(\max_{0\leqslant k\leqslant r}\left\{x+\left(\frac{r}{2}-k\right)t\right\}\right)^{-m}\\
&\leqslant 5^r C_2^{r/2}\left(\max_{0\leqslant k\leqslant r}\left\{\varphi\left(x+\left(\frac{r}{2}-k\right)t\right)\right\}\right)^{-r/2}t^r,
\end{aligned}$$

which implies (3.8).

Finally, we prove (3.8) for $rt/2<x<((r+1)/2)\,t$ and $r=2m-1$ with $m\in N$. By (3.8) with $r=2$ we have

$$\begin{aligned}
&\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left(\varphi\left(x+\sum_{j=1}^{r}y_j\right)\right)^{-r/2}dy_1\cdots dy_r\\
&\leqslant\left(\frac{16}{7}\right)^{r/2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left(\prod_{i=1}^{m-1}(x+y_{2i-1}+y_{2i})\right)^{-1}\\
&\qquad\times(x+y_r)^{-1/2}\,dy_1\cdots dy_r\\
&\leqslant\left(\frac{16}{7}\right)^{r/2}\left(\prod_{i=1}^{m-1}\left\{\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}(\varphi(x+y_{2i-1}+y_{2i}))^{-1}\,dy_{2i-1}\,dy_{2i}\right\}\right)\\
&\qquad\times\int_{-t/2}^{t/2}\frac{1}{\sqrt{x+y_r}}\,dy_r
\end{aligned}$$

$$\leqslant \left(\frac{16}{7}\right)^{r/2} \prod_{i=1}^{m-1} \{C_2 t^2 (\max_{0 \leqslant j \leqslant 2} \{\varphi(x+(1-j)\,t)\})^{-1}\}$$

$$\times 2\left(\sqrt{x+\frac{t}{2}}-\sqrt{x-\frac{t}{2}}\right)$$

$$\leqslant 2\left(\frac{16}{7}\right)^{r} C_2^{m-1} t^{r-1} (x+t)^{1-m} \frac{t}{\sqrt{x+\frac{t}{2}}+\sqrt{x-\frac{t}{2}}}$$

$$\leqslant 2\left(\frac{16}{7}\right)^{r} C_2^{(r-1)/2} t^r 2^{m-1+1/2} \left(\max_{0 \leqslant k \leqslant r} \left\{\varphi\left(x+\left(\frac{r}{2}-k\right)t\right)\right\}\right)^{1-m-1/2}$$

$$\leqslant 5^{r+1} C_2^{(r-1)/2} \left(\max_{0 \leqslant k \leqslant r} \left\{\varphi\left(x+\left(\frac{r}{2}-k\right)t\right)\right\}\right)^{-r/2} t^r.$$

Hence (3.8) also holds for the final case and the proof of Lemma 3.2 is complete.

The example $f(x)=x^3$ with $|B_n(f,x)-f(x)| \sim |x(1-x)(1-2x)\,n^{-2}|$ shows that the number $r-1$ of times of the combinations in (3.1) can not be reduced to $r-2$. The following theorem shows that the number r in (3.2) is also necessary.

THEOREM 3.3. *For any* $1<\alpha<2$ *there exist no functions* $\{h_{n,\alpha}(x)\}_{n \in N}$ *such that for* $f \in C[0,1]$

$$|K_n(f,x)-f(x)| \leqslant M h_{n,\alpha}(x) \Leftrightarrow \omega_2(f,t)=O(t^\alpha). \tag{3.9}$$

Proof of Theorem 3.3. Suppose that for some $1<\alpha<2$ there exist functions $\{h_{n,\alpha}(x)\}_{n \in N}$ that satisfy (3.9). Let $f(x)=x^i$, $i=1,2$. Then from the moments of the Kantorovich operators given in [10, Chapter 9] we have for $n \in N$, $x \in [0,1]$

$$|(1-2x)/n| \leqslant M_1 h_{n,\alpha}(x)$$

and

$$x(1-x)/(n+1)-1/(6(n+1)^2) \leqslant |K_n(t^2,x)-x^2| \leqslant M_2 h_{n,\alpha}(x).$$

Hence,

$$h_{n,\alpha}(x) \geqslant m/n$$

with a constant $m>0$ independent of $n \in N$ and $x \in [0,1]$.

Now let $g(x)=x^{\beta}$ with $1<\beta<\alpha$. We know from the saturation condition of the Kantorovich operators [12, 13] that

$$|K_n(g,x)-g(x)|\leqslant \|K_n g-g\|_\infty \leqslant M''/n\leqslant M''h_{n,\alpha}(x)/m.$$

However, $\omega_2(g,t)\neq(t^\alpha)$, which is a contradiction. Hence Theorem 3.3 holds.

4. Inverse Results for Derivatives

In this section we give the inverse part of (1.9) and (1.10). Some ideas of the proof are from [7, 10]. We use the combinations of operators to deal with higher levels of smoothness.

THEOREM 4.1. *Suppose that* $f\in C[0,1]$ *satisfies* $\omega_r(f,h)=O(h^\beta)$ *for certain* $\beta>0$, $r\in N$, $0<\alpha<r$. *Then we have*

$$\omega_r(f,h)=O(h^\alpha)\Leftrightarrow |B_n^{(r)}(f,x)|\leqslant L\left(\min\left\{n^2,\frac{n}{x(1-x)}\right\}\right)^{(r-\alpha)/2}. \quad (4.1)$$

Remark. The case $r=1,2$ were proved by Ditzian in [7, Theorem 2.1].

Proof of Theorem 4.1. By Theorem 2.2 we only need to prove the inverse part. Suppose that $\omega_r(f,h)\leqslant Lh^\beta$ and

$$|B_n^{(r)}(f,x)|\leqslant L\left(\min\left\{n^2,\frac{n}{x(1-x)}\right\}\right)^{(r-\alpha)/2}.$$

Let $h\in(0,1/(8r))$, $0<t\leqslant h$, $x\pm rt/2\in(0,1)$, $n\in N$. Then by (2.1) we have

$$\begin{aligned}
|\Delta_t^r f(x)| &\leqslant |\Delta_t^r(f-B_n(f,r-1,\cdot))(x)|\\
&\quad+\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\left|B_n^{(r)}\left(f,r-1,x+\sum_{j=1}^r y_j\right)\right|dy_1\cdots dy_r\\
&\leqslant 4^rM\omega_r(f,d(n,x,t))+\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}C^{r+1}L\\
&\quad\times\left(\min\left\{n^2,\frac{n}{\varphi(x+\sum_{j=1}^r y_j)}\right\}\right)^{(r-\alpha)/2}dy_1\cdots dy_r\\
&\leqslant 4^rM\omega_r(f,d(n,x,t))+C^{r+1}L\min\\
&\quad\times\left\{n^{r-\alpha}t^r,(C_r+1)\left(\max_{0\leqslant k\leqslant r}\left\{\sqrt{\frac{\varphi(x+(r/2-k)t)}{n}}\right\}\right)^{\alpha-r}t^r\right\}\\
&\leqslant 4^rM\omega_r(f,d(n,x,t))+C^{r+1}L(C_r+1)\,t^r(d(n,x,t))^{\alpha-r}.
\end{aligned}$$

Here we have used Lemma 3.2 and Hölder's inequality.

Now for any $\delta \in (0, 1/(8r))$, we choose $n \in N$ such that $d(n, x, t) \leqslant \delta < d(n-1, x, t) \leqslant 2d(n, x, t)$.

With this choice, we have

$$|\Delta_t^r f(x)| \leqslant M_1' \omega_r(f, \delta) + M_2' t^r \delta^{\alpha - r}$$

with the constants M_1', M_2' independent of x, t, h and δ. Hence

$$\omega_r(f, h) \leqslant M_1' \omega_r(f, \delta) + M_2' h^r \delta^{\alpha - r}.$$

Let $A = (2M_1' + 1)^{1/\alpha + 1/\beta}$, $\delta = h/A$. By induction we have for $k \in N$

$$\begin{aligned}\omega_r(f, h) &\leqslant M_1' \omega_r(f, h/A) + M_2' A^{r-\alpha} h^\alpha \leqslant \cdots \\ &\leqslant M_1'^k \omega_r(f, hA^{-k}) + \sum_{i=0}^{k-1} M_2' A^{r-\alpha} (M_1' A^{-\alpha})^i h^\alpha \\ &\leqslant M_1'^k L h^\beta A^{-k\beta} + 2M_2' A^{r-\alpha} h^\alpha.\end{aligned}$$

Thus, letting $k \to \infty$ we obtain

$$\omega_r(f, h) \leqslant 2M_2' A^{r-\alpha} h^\alpha.$$

The proof of Theorem 4.1 is complete.

Remark. We conjecture that the assumption $\omega_r(f, h) = O(h^\beta)$ in Theorem 4.1 can be dropped.

For the Kantorovich operators, our result on derivatives is simpler.

THEOREM 4.2. *Let $f \in C[0, 1]$, $r \in N$, $0 < \alpha < r$. Then we have*

$$\omega_r(f, h) = O(h^\alpha) \Leftrightarrow |K_n^{(r)}(f, x)| \leqslant M \left(\min \left\{ n^2, \frac{n}{x(1-x)} \right\} \right)^{(r-\alpha)/2}. \tag{4.2}$$

The proof of Theorem 4.2 follows using Theorem 4.1 and the method in [7, p. 30]. We omit it here.

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